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Veneziano Model for $\pi\pi \to \pi S$, Where S Has Arbitrary Spin and Parity*

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The Veneziano model is applied to the reaction $\pi\pi \to \pi S$, where S is any particle of spin S, isospin 0 or 1. For particles with natural parity, $P=(-1)^S$, we require that our amplitude not be coupled to any odd daughters. This fixes the relative magnitudes of the helicity amplitudes and constrains the particles $S(\omega,A_2,\cdots)$ to lie on a trajectory which is parallel to the degenerate ρ and f trajectories. For particles with unnatural parity, $P=-(-1)^S$, we demand that the amplitude satisfy the Adler consistency condition. This gives mass restrictions on the sequence of particles π, A_1, $\cdots$.

I. INTRODUCTION

RECENTLY Veneziano[1] has proposed a simple and elegant form for the scattering amplitude for the process $\pi\pi \to \pi\omega$. The expression for the scattering amplitude in terms of β functions satisfies crossing symmetry and Regge asymptotic behavior. He also noted that a restriction on the sum of the trajectories in the form $\alpha(s_{12})+\alpha(s_{23})+\alpha(s_{31})=2$ resulted in the decoupling of "odd daughters" and gave the intercept for the exchanged ρ trajectory in very good agreement with experiments. While many ambiguities of the simple Veneziano form have been debated in the literature,[2] it is of considerable interest to extend his considerations to other reactions[3] and explore the physical consequences. In this paper we investigate the consequences of such a generalization to the case $\pi\pi \to \pi S$, where S is a particle of arbitrary spin S.

In Sec. II, we treat the case when S has natural parity, with $C=P=(-1)^S$. We show that there exists a simple amplitude satisfying crossing symmetry and Regge asymptotic behavior, in which, under the condition $\alpha(s_{12})+\alpha(s_{23})+\alpha(s_{31})=S+1$, the odd daughters are decoupled. This generalizes the result of Ademollo *et al.*[4] This condition is simultaneously satisfied by all resonances on the degenerate ρ-f trajectory. In Sec. III, we treat the case of unnatural parity, $P=-(-1)^S$. Here we impose the Adler consistency condition[5] that the amplitude should vanish when any pion four-momentum vanishes and discuss the restrictions on the trajectory on which the unnatural particles lie. Section IV is devoted to the summary of results. Finally, in the Appendix, we write out the amplitudes in a few simple cases.

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[4] M. Ademollo, H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Rev. **176**, 1904 (1968).

[5] S. L. Adler, Phys. Rev. **137**, B1022 (1965).

II. NATURAL PARITY

The observed particles with natural parity, $P=(-1)^S$, have $C=P$, which is consistent with quark-antiquark states. Hence, those particles which are coupled to three pions have $G=(-1)^{I+S}=-1$. Thus, the particle S has even (odd) isospin I when S is odd (even). We consider only isospins 0 and 1, so that the particle S has the quantum numbers of the ω or the A_2. Similarly, we assume that the exchanged particles have isospins 0 or 1, and they therefore have the quantum numbers of the ρ or the f_0.

A term of the amplitude for the reaction $\pi\pi \to \pi S$, S with natural parity, can be written

$$A(s_{12},s_{23},s_{31}) = \text{const}\times[I_{123}\epsilon_{\mu\alpha\beta\gamma}p_{1\alpha}p_{2\beta}p_{3\gamma}S_{\mu\cdots}(p_3)^L(p_1)^M V(s_{12},s_{23}) + \text{permutations on (123)}]. \quad (2.1)$$

The isospin factor I_{123} is discussed below [see Eqs. (2.6) and (2.7)]. All pion four-momenta are taken to be incoming (see Fig. 1). $S_{\mu\cdots}$ is the polarization tensor for particle S. It is symmetric in its S indices, traceless and orthogonal to p_S, i.e.,

$$S_{\mu\cdots\sigma}p_{S\sigma} = -S_{\mu\cdots\sigma}(p_1+p_2+p_3)_\sigma = 0. \quad (2.2)$$

In (2.1), $S_{\mu\cdots}(p_3)^L(p_1)^M$ is understood to mean the contraction of the polarization tensor $S_{\mu\cdots}$ with L factors of p_3 and M factors of p_1. Thus, L and M must satisfy $L+M=S-1$. The orthogonality condition (2.2) together with the four-momentum conservation equation $p_1+p_2+p_3+p_S=0$ means that in the contraction

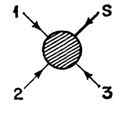

FIG. 1. Diagram for the reaction $\pi\pi \to \pi S$. The pion four-momenta p_1, p_2, p_3 and the four-momentum of S, p_S, are all incoming, $p_1+p_2+p_3+p_S=0$. The invariants s_{ij} are defined to be $s_{ij}=-(p_i+p_j)^2$.

of $S_{\mu}\ldots$ with powers of the momenta, only two of the latter are independent; we have chosen to use p_1 and p_3. By summing over all permutations in (2.1), we automatically satisfy crossing symmetry. It is necessary, however, to consider the restrictions on $V(s_{12},s_{23})$ from asymptotic behavior.

For this purpose, consider the limit as $s_{12}\to\infty$ with s_{23} fixed. Since the kinematic factor multiplying $V(s_{12},s_{23})$ goes as s_{12}^{L+1} in this limit, $V(s_{12},s_{23})$ should have the asymptotic behavior

$$V(s_{12},s_{23})\to s_{12}^{\alpha_{23}-L-1}\begin{pmatrix} s_{12}\to\infty \\ s_{23}\text{ fixed}\end{pmatrix}, \tag{2.3}$$

where $\alpha_{23}=\alpha_0+\alpha' s_{23}$ is the exchanged ρ-f trajectory. The other relevant term $V(s_{23},s_{12})$ is generated in the permutation where p_1 and p_3 are interchanged. The kinematic factor of this term goes as s_{12}^{M+1}; hence $V(s_{23},s_{12})$ should have the asymptotic behavior

$$V(s_{23},s_{12})\to s_{12}^{\alpha_{23}-M-1}\begin{pmatrix} s_{12}\to\infty \\ s_{23}\text{ fixed}\end{pmatrix}. \tag{2.4}$$

These considerations suggest for $V(s_{12},s_{23})$ the modified Veneziano form[6]

$$\frac{\Gamma(S-L-\alpha_{12})\Gamma(S-M-\alpha_{23})}{\Gamma(S+1-\alpha_{12}-\alpha_{23})}. \tag{2.5}$$

This is the Euler β function $B(S-L-\alpha_{12},\, S-M-\alpha_{23})$.

The isospin factor I_{123} is uniquely determined. If S is odd, i.e., $I_S=0$,

$$I_{123}=\boldsymbol{\eta}_1\cdot(\boldsymbol{\eta}_2\times\boldsymbol{\eta}_3), \tag{2.6}$$

where $\boldsymbol{\eta}_i$ denotes the isotopic-spin vector of the particle i. If S is even, i.e., $I_S=1$,

$$I_{123}=(\boldsymbol{\eta}_1\cdot\boldsymbol{\eta}_2)(\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_S)+(\boldsymbol{\eta}_1\cdot\boldsymbol{\eta}_S)(\boldsymbol{\eta}_2\cdot\boldsymbol{\eta}_3)-(\boldsymbol{\eta}_1\cdot\boldsymbol{\eta}_3)(\boldsymbol{\eta}_2\cdot\boldsymbol{\eta}_S). \tag{2.7}$$

The particular form of I_{123} in (2.7) results from our requirement that there are no exchanged particles with $I=2$. Written in terms of projection operators, I_{123} in (2.7) is $3P_0+2P_1$, where P_0, P_1 are the 12-channel $I=0$, 1 projection operators.[7]

Now consider the amplitude

$$A^S=\text{const}\times\left[I_{123}\epsilon_{\mu\alpha\beta\gamma}p_{1\alpha}p_{2\beta}p_{3\gamma}S_{\mu}\ldots\times\int_0^1 dx\frac{[xp_1-(1-x)p_3]^{S-1}}{x^{\alpha_{12}}(1-x)^{\alpha_{23}}}+\text{permutations}\right], \tag{2.8}$$

which is a particular combination of terms from (2.1) and (2.5). $S_{\mu}\ldots[xp_1-(1-x)p_3]^{S-1}$ is again understood to mean the contraction of $S_{\mu}\ldots$ with the appropriate factors of p_1 and p_3 after the expansion of the binomial has been performed. Thus, for example, if $S=1$,

$$\int_0^1 dx\, x^{-\alpha_{12}}(1-x)^{-\alpha_{23}}=B(1-\alpha_{12},\,1-\alpha_{23}), \tag{2.9}$$

which is the Veneziano form for $\pi\pi\to\pi\omega$. If $S=3$, we have

$$\begin{aligned}S_{\mu\rho\sigma}\int_0^1 dx\frac{[xp_1-(1-x)p_3]^2}{x^{\alpha_{12}}(1-x)^{\alpha_{23}}}
&=S_{\mu\rho\sigma}[p_{1\rho}p_{1\sigma}B(3-\alpha_{12},\,1-\alpha_{23})\\
&\quad-2p_{1\rho}p_{3\sigma}B(2-\alpha_{12},\,2-\alpha_{23})\\
&\quad+p_{3\rho}p_{3\sigma}B(1-\alpha_{12},\,3-\alpha_{23})],\end{aligned} \tag{2.10}$$

From (2.8) the part of A^S that contributes to the poles in, say, the 12 channel, is given by

$$A^S=\begin{Bmatrix}4\lambda_S\\ 6\lambda_S\\ 2\lambda_S'\end{Bmatrix}\epsilon_{\mu\alpha\beta\gamma}p_{1\alpha}p_{2\beta}p_{3\gamma}S_{\mu}\ldots\times\left[\int_0^1 dx\frac{[xp_1-(1-x)p_3]^{S-1}}{x^{\alpha_{12}}(1-x)^{\alpha_{23}}}-(-1)^{I_{12}}\int_0^1 dx\frac{[xp_2-(1-x)p_3]^{S-1}}{x^{\alpha_{12}}(1-x)^{\alpha_{31}}}\right], \tag{2.11}$$

for the cases

$$S=\text{even},\ \begin{pmatrix}I_{12}=1\\ I_{12}=0\end{pmatrix},$$

$$S=\text{odd},\ I_{12}=1,$$

where I_{12} is the isospin of channel 12. λ_S and λ_S' are arbitrary constants.

"Odd daughters"[8] of the 12 channel occur in $I_{12}=1$ (0) when $\alpha_{12}=$ even (odd). We shall now prove that in

[6] The form for $V(s,t)$ given in (2.5) is not unique. One can add an integer j to the argument of each Γ function in (2.5). One could also add terms with nonleading asymptotic behavior. We will consider only the simplest Veneziano form, the one which has the maximum allowed asymptotic behavior and which permits coupling to all the allowed particles on the internal trajectories.

[7] For convenience, we note that permutations of I_{123} in (2.9) can be written as $I_{123}=I_{321}=3P_0+2P_1$, $I_{231}=I_{132}=2P_2-P_0$, and $I_{312}=I_{213}=3P_0-2P_1$, where P_0, P_1, and P_2 are the 12-channel $I=0$, 1, 2 isospin projection operators, respectively.

[8] The Veneziano model has an infinite number of parallel trajectories with intercepts spaced in unit intervals below the leading trajectory α. The particles on the trajectories which have intercepts $2n+1$ units below the leading trajectory are called the "odd daughters." These daughters should not be confused with the Freedman-Wang daughters.

(2.8) the odd daughters are decoupled if

$$\alpha_{12}+\alpha_{23}+\alpha_{31}=S+1 \tag{2.12}$$

by showing that the poles in α_{12} in (2.11) vanish when $(-1)^{\alpha_{12}+I_{12}}=-1$. For this purpose, consider the second term (2.11). By using the momentum conservation equation,

$$\int_0^1 dx\frac{[xp_2-(1-x)p_3]^{S-1}}{x^{\alpha_{12}}(1-x)^{\alpha_{31}}} =(-1)^{S-1}\int_0^1 dx\frac{(xp_1+p_3)^{S-1}}{x^{\alpha_{12}}(1-x)^{\alpha_{31}}}. \tag{2.13}$$

After making the binomial expansion of the numerator of (2.13), we consider a typical term

$$(-1)^{S-1}p_1{}^{S-1-j}p_3{}^j\int_0^1 dx\frac{x^{S-1-j}}{x^{\alpha_{12}}(1-x)^{\alpha_{31}}}. \tag{2.14}$$

Now from the properties of the Γ functions one can easily prove that

$$\int_0^1 dx\, x^{-\alpha}(1-x)^{-\beta}=(-1)^{\alpha+1} \times\int_0^1 dx\, x^{-\alpha}(1-x)^{-2+\alpha+\beta}, \tag{2.15}$$

when α is an integer. Using this property, (2.14) can be transformed to

$$\begin{aligned}(-1)^{S-1}p_1{}^{S-1-j}p_3{}^j&\int_0^1 dx\, x^{S-1-j-\alpha_{12}}(1-x)^{-\alpha_{23}}\\ &=(-1)^{S-1}p_1{}^{S-1-j}p_3{}^j(-1)^{S-j-\alpha_{12}}\\ &\quad\times\int_0^1 dx\, x^{S-1-j-\alpha_{12}}(1-x)^{(\alpha_{12}+\alpha_{31}-S-1+j)}, \qquad (2.16)\\ &=(-1)^{\alpha_{12}+1}p_1{}^{S-1-j}(-p_3)^j\int_0^1\frac{x^{S-1-j}(1-x)^j}{x^{\alpha_{12}}(1-x)^{\alpha_{23}}}, \qquad (2.17)\end{aligned}$$

provided $\alpha_{12}+\alpha_{23}+\alpha_{31}=S+1$. Combining all the terms in the expansion, the second term, therefore, can be written

$$(-1)^{\alpha_{12}+1}\int_0^1 dx\frac{[xp_1-(1-x)p_3]^{S-1}}{x^{\alpha_{12}}(1-x)^{\alpha_{23}}}. \tag{2.18}$$

Thus the total 12-channel pole contribution is proportional to $1+(-1)^{I_{12}+\alpha_{12}}$ and hence the stated result follows.

III. UNNATURAL PARITY

We now consider the case where the particle S has parity $(-1)^{S+1}$. The physically interesting particles in this case have the quantum numbers of π and A_1. For the sake of completeness in what follows, we will consider S to have both isospins 0 and 1.

The amplitude in the present case will have the general form

$$\begin{aligned}A(s_{12},s_{23},s_{31})&\\ =\text{const}\times[&I_{123}S\cdots(p_3)^L(p_1)^M V(s_{12},s_{23})\\ &+\text{permutations}]. \qquad (3.1)\end{aligned}$$

The notation is the same as in Sec. II, but with the condition $L+M=S$. If we use the symmetry[7] $I_{123}=(-1)^{I_S+1}I_{321}$, we can write the amplitude A in (3.1) in the form

$$\begin{aligned}A=\text{const}\times S\cdots\{&I_{123}[p_3{}^Lp_1{}^MV(s_{12},s_{23})\\ &-(-1)^{I_S}p_1{}^Lp_3{}^MV(s_{23},s_{12})]+I_{231}[p_1{}^Lp_2{}^MV(s_{23},s_{31})\\ &-(-1)^{I_S}p_2{}^Lp_1{}^MV(s_{31},s_{23})]+I_{312}[p_2{}^Lp_3{}^MV(s_{31},s_{12})\\ &-(-1)^{I_S}p_3{}^Lp_2{}^MV(s_{12},s_{31})]\}. \qquad (3.2)\end{aligned}$$

Note that we need only consider forms with $L\geq M$, since up to an over-all sign, the amplitude generated by permutations from $p_3{}^Lp_1{}^MV(s_{12},s_{23})$ is the same as that generated from $p_1{}^Lp_3{}^MV(s_{23},s_{12})$.

By requiring the amplitude A to have the proper asymptotic behavior, from considerations analogous to (2.3) and (2.4), the function $V(s_{12},s_{23})$ is constrained to be a sum of terms of the form

$$\frac{\Gamma(m-\alpha_{12})\Gamma(n-\alpha_{23})}{\Gamma(m+n-p-\alpha_{12}-\alpha_{23})}, \tag{3.3}$$

with $m\geq M+p$, $n\geq L+p$, and $p=\text{integer}\geq 0$. The last ensures that the residues of the poles of (3.3) are polynomials. Further, if all the poles are to be either on the leading ρ-f trajectory, or its daughters, then m and n should be integers. Finally, the conditions $m\geq 1$, $n\geq 1$ must hold in order to avoid poles at $\alpha_{ij}=0$, which would be the "P' ghost" on the ρ-f trajectory at $s_{ij}\simeq-0.5$ GeV^2. For later use, we note that (3.3) vanishes if and only if

$$\alpha_{12}+\alpha_{23}+\alpha_{31}-m-n+p=\text{integer}\geq 0$$

and

$$\alpha_{12}-m\neq\text{integer}\geq 0\,,\quad \alpha_{23}-n\neq\text{integer}\geq 0\,. \tag{3.4}$$

We now insist that the amplitudes satisfy the Adler consistency condition,[5,9] namely, that the amplitudes vanish when a pion four-momentum vanishes. In the natural-parity case, the amplitude automatically vanishes because of the kinematic factor $\epsilon_{\mu\alpha\beta\gamma}p_{1\alpha}p_{2\beta}p_{3\gamma}$. Here, however, we will find that in order to satisfy the Adler condition (a) certain leading amplitudes must

[9] C. Lovelace, Phys. Letters **28B**, 265 (1968).

be absent or (b) the particles S must lie on a well-defined family of trajectories.[10,11]

For the sake of definiteness, let us consider the limit $p_2 \to 0$. Since we generate crossing-symmetric amplitudes, the same conclusions will be valid if we let p_1 or p_3 go to zero. In the limit $p_2 \to 0$,

$$\alpha_{12}=\alpha_{23}=\alpha_\rho(m_\pi^2)\,, \quad \alpha_{31}=\alpha_\rho(m_S^2)\,, \qquad (3.5)$$

and $S_{\ldots\sigma}p_{1\sigma}=-S_{\ldots\sigma}p_{3\sigma}$, because of (2.2). If we make the arbitrary assumption that the amplitude contains only leading terms, i.e., in (3.2), the functions V are of the form (3.3), where m, n, and p satisfy the equalities $m=M+p$ and $n=L+p$, then in order to satisfy the Adler condition we note the following: The coefficient of I_{123} [in (3.2)] vanishes only if

$$(-1)^{I_S}=(-1)^S \quad \text{or} \quad V(\alpha(m_\pi^2),\alpha(m_\pi^2))=0\,. \qquad (3.6)$$

The coefficients of I_{231} and I_{312} vanish only if

$$V(\alpha(m_\pi^2),\alpha(m_S^2))=0 \quad \text{or} \quad M\neq 0\,. \qquad (3.7)$$

We first take the case $S=0$. Here $M=L=0$, so $m=n=p\geq 1$. If $I_S=1$, as in the case when S is the pion, (3.6) implies that we must have $V(\alpha(m_\pi^2),\alpha(m_\pi^2))=0$, i.e., from (3.4),

$$2\alpha(m_\pi^2)=\text{integer}\geq p\geq 1\,. \qquad (3.8)$$

Experimentally, $2\alpha(m_\pi^2)\simeq 1$. Lovelace[9] first observed that this ensured the Adler condition for a π-π amplitude of the form (3.3), with $m=n=p=1$. Condition (3.7), $V(\alpha(m_\pi^2),\alpha(m_S^2))=0$, is then satisfied for any 0^- particle belonging to the same family as the pion. This family of 0^- particles will have masses squared which are given by

$$m_{0^-}^2=m_\pi^2+R\times(\alpha')^{-1}\,, \quad R=1,\,2,\cdots.$$

We now consider $S=1$; here $L=1$, $M=0$, so that $m=p\geq 1$, $n=1+p$. In order to have the coefficient of I_{123} in (3.2) vanish [condition (3.6)], we need either $I_S=1$ or $2\alpha(m_\pi^2)\geq 1+p$. Since $2\alpha(m_\pi^2)=1$, the second possibility, which requires $p=0$, is ruled out by $m=p\geq 1$. Thus condition (3.6) cannot be satisfied by the leading amplitude for an $I=0$, $S^P=1^+$ particle.[12] For an $I=1$, $S^P=1^+$ particle—call it A—condition (3.6) is satisfied, and condition (3.7) is satisfied if $\alpha(m_\pi^2)+\alpha(M_A^2)=\text{integer}\geq S+1=2$. The integer p must be chosen greater than or equal to 1 in order to avoid the P' ghost. Since $\alpha(m_\pi^2)=\frac{1}{2}$, this last equation can be written

$$\alpha(M_A^2)-\alpha(m_\pi^2)=\text{integer}\geq 1\,. \qquad (3.9)$$

Hence, the Adler condition is satisfied if the A is on a trajectory parallel to the ρ-f trajectory which is either the π trajectory or one of its daughters. The observed $A_1(1070)$ is in fact on a trajectory with the pion, which has the canonical slope$\simeq$1 GeV^{-2}. The condition $2\alpha(m_\pi^2)=1$ implies that the π-A_1 trajectory is one-half unit below the ρ-f trajectory. Furthermore, we obtain the famous relation $m_{A_1}^2+m_\pi^2=2m_\rho^2$.

For higher spins, $S\geq 2$, we first notice that condition (3.6) cannot be satisfied for $(-1)^{I_S}=(-1)^{S+1}$. We can, however, make the coefficient of I_{123} vanish as $p_2\to 0$ by taking a combination of terms of the form (3.2), where M varies over the range $0\leq M\leq \frac{1}{2}S$. For L, $M\neq 0$, the last four terms in (3.2) vanish since they have a coefficient p_2^L or p_2^M. If $M=0$, however, we must satisfy (3.7):

$$\alpha(m_\pi^2)+\alpha(M_S^2)\geq S+1\,.$$

This condition requires the particle S to be on the π-A_1 trajectory or one of its daughters.

IV. SUMMARY

We have given a general method of constructing crossing-symmetric Veneziano-type amplitudes for the reaction $\pi\pi\to\pi S$, where S is a particle of arbitrary spin S and parity $\pm(-1)^S$. When S belongs to the natural-parity sequence, we have found a solution in which all the odd daughters of the ρ-f trajectory are decoupled, provided the exchanged trajectories satisfy the condition $\alpha_{12}+\alpha_{23}+\alpha_{31}=S+1$, i.e.,

$$(M_S^2+3m_\pi^2)\alpha_\rho'=S+1-3\alpha_\rho(0)\,. \qquad (4.1)$$

The condition (4.1) implies that the natural-parity particles lie on a trajectory parallel to the ρ-f trajectory with intercept $3\alpha_\rho(0)-1+3\alpha_\rho'm_\pi^2$. This is in fact well satisfied by the ω-A_2 trajectory.

In the unnatural-parity case, we impose the Adler condition and find that the unnatural-parity particles must lie on a trajectory α_U which satisfies

$$\alpha_U(M_S^2)\leq\alpha_\rho(M_S^2)+\alpha_\rho(m_\pi^2)-1\,. \qquad (4.2)$$

If we consider the pion,[9] $\alpha_U(m_\pi^2)=0$ and hence $2\alpha_\rho(m_\pi^2)=\text{integer}\geq 1$; empirically the equality holds. Thus $\alpha_U(M_S^2)\leq\alpha_\rho(M_S^2)-\frac{1}{2}$. This means that the trajectory α_U must be a member of the family of trajectories whose parent is the pion trajectory which must be parallel to, and lie one-half unit below the ρ-f trajectory. Similar results are obtained by Ademollo *et al.*[10]

Clearly, the above results on the trajectories are in good agreement with experimental data, as far as they exist. It is not clear, however, why this should be so. In the natural-parity case, the mystery is why odd

[10] M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters **22**, 83 (1969).

[11] We would like to note here that for particles S with isospin 1 (0) and with even (odd) spin, we can construct an amplitude, similar to Eq. (2.7), which decouples all odd daughters,

$$A\sim\left[I_{123}S\ldots\int_0^1 dx\frac{[xp_1-(1-x)p_3]^S}{x^{\alpha_{12}+1}(1-x)^{\alpha_{23}}}+\text{permutations}\right],$$

provided $\alpha_{12}+\alpha_{23}+\alpha_{31}=S-1$. This restriction, however, leads to "ghosts." The leading ρ and f_0 trajectories have the form $\alpha_{12}\simeq\frac{1}{2}+s_{12}/2m_\rho^2$. Hence in the limit $m_\pi^2=0$, the particles with $S=0$, 1, 2 will have masses (squared) equal to $-5m_\rho^2$, $-3m_\rho^2$, and $-m_\rho^2$, respectively. Besides, the amplitude does not satisfy the Adler condition.

[12] According to the latest Particle Data Group [N. Barash-Schmidt *et al.*, Rev. Mod. Phys. **41**, 109 (1969)], there is no particle known with $I^G=0^-$, $S^P=1^+$.

daughters should be required to be absent from the amplitude under consideration, and in particular through the condition (4.1) on the ρ-f trajectory rather than by subtraction of nonleading terms, *à la* Mandelstam.[13] It should be noted that since "odd daughters" are present in $\pi\pi$ scattering, their absence on $\pi\pi \to \pi S$ would imply the vanishing of their coupling to πS, $S=\omega$, A_2, $\cdots$. In the unnatural-parity case, the Adler condition is a well-established principle, but it is not clear why it should be satisfied by the leading Veneziano form alone. There is the further mystery of why the lack of Pomeranchon exchange in the amplitude was not serious.

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APPENDIX

The purpose of this Appendix is to illustrate the application of our general results in a few simple cases.

A. $\pi\pi \to \pi A_2$

From (2.1) ,the 1-2 channel poles are given by

$$A^2=\begin{pmatrix}4\lambda_2\\6\lambda_2\end{pmatrix}\epsilon_{\mu\alpha\beta\gamma}p_{1\alpha}p_{2\beta}p_{3\gamma}S_{\mu\nu}\left[\left(p_{1\nu}\frac{\Gamma(2-\alpha_{12})\Gamma(1-\alpha_{23})}{\Gamma(3-\alpha_{12}-\alpha_{23})}-p_{3\nu}\frac{\Gamma(1-\alpha_{12})\Gamma(2-\alpha_{23})}{\Gamma(3-\alpha_{12}-\alpha_{23})}\right)\right.$$
$$\left.\pm\left(p_{2\nu}\frac{\Gamma(2-\alpha_{12})\Gamma(1-\alpha_{31})}{\Gamma(3-\alpha_{12}-\alpha_{31})}-p_{3\nu}\frac{\Gamma(1-\alpha_{12})\Gamma(2-\alpha_{13})}{\Gamma(3-\alpha_{12}-\alpha_{23})}\right)\right]. \quad \text{(A1)}$$

Hence, the residues of the ρ and f_0 pole are, respectively, given by

$$+8\lambda_2\epsilon_{\mu\alpha\beta\gamma}p_{1\alpha}p_{2\beta}p_{3\gamma}S_{\mu\nu}p_{3\nu}(\alpha')^{-1} \quad \text{(A2)}$$

and

$$-6\lambda_2\epsilon_{\mu\alpha\beta\gamma}p_{1\alpha}p_{2\beta}p_{3\gamma}S_{\mu\nu}[p_{1\nu}-p_{2\nu}+p_{3\nu}(\alpha_{31}-\alpha_{23})](\alpha')^{-1}, \quad \text{(A3)}$$

with

$$\alpha_{31}-\alpha_{23}=\alpha'(s_{31}-s_{23})=2\alpha' p_3\cdot(p_2-p_1).$$

If we write the elementary $\rho\pi\pi$ and $A_2\rho\pi$ couplings in momentum space in the form

$$g_{\rho\pi\pi}(p_1-p_2)_\mu\rho_\mu(\boldsymbol{\eta}_1\times\boldsymbol{\eta}_2)\cdot\boldsymbol{\eta}_\rho \quad \text{(A4)}$$

and

$$g_{A_2\rho\pi}\epsilon_{\mu\alpha\beta\gamma}S_{\mu\nu}p_{3\nu}p_{3\alpha}p_{S\beta}\rho_\gamma\boldsymbol{\eta}_\rho\cdot(\boldsymbol{\eta}_S\times\boldsymbol{\eta}_3), \quad \text{(A5)}$$

where ρ_μ is the polarization vector of ρ, we obtain for the residue of the ρ pole

$$4g_{A_2\rho\pi}g_{\rho\pi\pi}\epsilon_{\mu\alpha\beta\gamma}p_{1\alpha}p_{2\beta}p_{3\gamma}S_{\mu\nu}p_{3\nu}. \quad \text{(A6)}$$

Hence

$$\lambda_2=+\tfrac{1}{2}\alpha' g_{A_2\rho\pi}g_{\rho\pi\pi}. \quad \text{(A7)}$$

Thus λ_2 can be determined from the known decay rates of $A_2\to\rho\pi$ and $\rho\to\pi\pi$. The products of the couplings $S\pi\pi$ and $SA_2\pi$ are then completely determined. Thus we note that in (A3) products of couplings of $f\pi\pi$ and $fA_2\pi$ appear. Equation (A3) contains the two independent couplings $A_2f\pi$ in a definite ratio.

B. $\pi\pi \to \pi A_1$

From the general considerations in Sec. III, there are two independent amplitudes for $\pi\pi\to\pi A_1$. A general amplitude can be constructed in the form

$$A^1=S_\mu\lambda_1\left[I_{123}p_{1\mu}\frac{\Gamma(2-\alpha_{12})\Gamma(1-\alpha_{23})}{\Gamma(2-\alpha_{12}-\alpha_{23})}+\text{permutations}\right]+S_\mu\lambda_1'\left[I_{123}p_{2\mu}\frac{\Gamma(1-\alpha_{12})\Gamma(1-\alpha_{23})}{\Gamma(2-\alpha_{12}-\alpha_{23})}+\text{permutations}\right]. \quad \text{(A8)}$$

The part of the amplitude which contains 12-channel poles is given by

$$A^1=S_\mu\begin{pmatrix}2\lambda_1\\3\lambda_1\end{pmatrix}\left[\left(p_{1\mu}\frac{\Gamma(2-\alpha_{12})\Gamma(1-\alpha_{23})}{\Gamma(2-\alpha_{12}-\alpha_{23})}+p_{3\mu}\frac{\Gamma(2-\alpha_{23})\Gamma(1-\alpha_{12})}{\Gamma(2-\alpha_{12}-\alpha_{23})}\right)\right.$$
$$\left.\mp\left(p_{3\mu}\frac{\Gamma(2-\alpha_{31})\Gamma(1-\alpha_{12})}{\Gamma(2-\alpha_{31}-\alpha_{12})}+p_{2\mu}\frac{\Gamma(2-\alpha_{12})\Gamma(1-\alpha_{31})}{\Gamma(2-\alpha_{12}-\alpha_{31})}\right)\right]$$
$$+\begin{pmatrix}4\lambda_1'\\6\lambda_1'\end{pmatrix}\left[p_{2\mu}\frac{\Gamma(1-\alpha_{12})\Gamma(1-\alpha_{23})}{\Gamma(2-\alpha_{12}-\alpha_{23})}\mp p_{1\mu}\frac{\Gamma(1-\alpha_{12})\Gamma(1-\alpha_{31})}{\Gamma(2-\alpha_{12}-\alpha_{31})}\right]. \quad \text{(A9)}$$

[13] Mandelstam (Ref. 2) has discussed the possibility of decoupling odd daughters in the case of spinless particles without any restrictions on the trajectories. It is to be seen whether this procedure can be generalized to the case of arbitrary spins and also whether the Adler condition can be maintained.

For the ρ pole, the residue is given by

$$-\frac{2\lambda_1}{\alpha'}S_\mu p_{3\mu}(\alpha_{23}-\alpha_{31})-\frac{4\lambda_1'}{\alpha'}S_\mu(p_2-p_1)_\mu=+4\lambda_1 S_\mu p_{3\mu}p_3\cdot(p_2-p_1)-\frac{4\lambda_1'}{\alpha'}S_\mu(p_2-p_1)_\mu. \tag{A10}$$

For the two independent $A\rho\pi$ couplings, we take

$$(g_1S_\mu\rho_\mu\times g_2S_\mu p_{3\mu}p_{S\nu}\rho_\nu)(\boldsymbol{\eta}_\rho\cdot\boldsymbol{\eta}_S\times\boldsymbol{\eta}_3). \tag{A11}$$

Using (A11) and (A4), we calculate the contribution of the ρ pole to $\pi\pi\to\pi A_1$ and find for the residue of the ρ pole

$$+2g_1g_{\rho\pi\pi}S_\mu(p_2-p_1)_\mu+2g_2g_{\rho\pi\pi}S_\mu p_{3\mu}p_3\cdot(p_2-p_1). \tag{A12}$$

Comparing (A10) and (A12), we see that

$$\lambda_1=\tfrac{1}{2}g_2g_{\rho\pi\pi},\quad \lambda_1'=-\tfrac{1}{2}\alpha' g_1g_{\rho\pi\pi}. \tag{A13}$$

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Pion Gauge Invariance and Low-Energy Theorems*

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Zero-mass pion theories invariant under c-number translations ("chiral transformations") of the pion field are studied in a general framework. The operator which induces the chiral transformation is defined in Fock space (in which it is not unitary) and in von Neumann's infinite-tensor-product space (in which it is unitary). The transformed (noninvariant) Fock-space vacuum is recognized as a coherent state in the tensor-product space. The generator of the chiral transformation—a constant of the motion in gauge-invariant theories—is diagonalized, and its eigenvectors, the "chiral states," are employed in one of two derivations of a low-energy theorem for zero-mass pion emission and absorption, assuming gauge invariance of the theory. The other method of derivation is also used to rederive the electromagnetic gauge conditions. Then Lagrangian models (gradient-coupling, c-number, and operator theory) are studied in which the invariance is realized provided the current is suitably restricted. Implications of the low-energy theorem are checked (exactly for the c-number theory, in lowest-order perturbation theory for the operator theory). A larger class of models is then considered in which, it is shown, the complicated set of transformations under which the Lagrangian is invariant reduce, by virtue of the field equations and the asymptotic condition, to a simple pion translation when expressed in terms of the asymptotic fields, and hence obey the supposition of our theorem, which we again check in lowest-order perturbation theory.

1. INTRODUCTION

In quantum electrodynamics, the invariance of the vector potential against local guage transformations $A_{\mu\text{ in (out)}}(x)\to A_{\mu\text{ in (out)}}(x)+\partial_\mu\Lambda(x)$ is necessary because only then is the theory a Lorentz-invariant description of zero-mass (spin-one) particles.[1] A gauge principal for pion interactions, $\phi_{\text{in (out)}}\to\phi_{\text{in (out)}}+c$, however, is apparently neither natural nor necessary. No classical limit exists (as in quantum electrodynamics) which guides one to such an invariance; moreover the invariance requires pions of zero mass and is therefore physically interesting only when it is broken. However, many theories of current interest may be pion gauge-invariant in the limit of vanishing pion mass. We have in mind the various phenomenological Lagrangians of the past few years, which are invariant under a set of transformations (of the interacting fields) which contains $\phi\to\phi+c$. Thus, we expect amplitudes calculated from such Lagrangians to obey pion gauge conditions in the limit of zero-pion mass. Since the Lagrangians (which appear to incorporate the current algebra results) include many ingredients beside pion gauge invariance in the zero-mass limit, it is interesting to determine which if any of their predictions are due solely to the pion gauge conditions. The Adler[2] consistency condition, for example, requires that the πN forward scattering amplitude vanishes when one of the pion's four-momentum goes to zero. This result is essentially based on the hypothesis of partially con-

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